Mass-spring-damper systems

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1 Components

Here are the basic characteristics of the components in mechanics.

1.1 Mass

When a mass is moving, it's force can be calculated using Newton's second law directly

$$f = ma = m\dot{v} = m\ddot{x} \tag{1}$$

Figure 1: Mass

1.2 Spring

A spring has the ability to extend and compress, and it's force depends on it's level of compression or expansion.

$$f = k(x_2 - x_1) \tag{2}$$

It is assumed, that the spring is massless, and the force in either end equals the force of the others, but in the opposite direction (Newton's 3rd law).

Figure 2: Spring

1.3 Damper

A damper is a device, that has a force dependant on velocity. There are two kinds of dampers: The piston type, that can be connected between masses, and the friction type, that is always with reference to the stationary ground.

$$f = b(\dot{x}_2 - \dot{x}_1) \tag{3}$$

or for fricition

$$f = b\dot{x} \tag{4}$$

Either of these are approximations and emulates so called *viscous friction*, and is a good approximation if the surface is lubricated. It does not take into account, the extra force needed when first starting to move a box ("dry friction" and "sticktion").

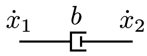

Figure 3: Damper

2 Example system

A translational mass-spring-damper system is given in Figure 4. We wish to derive the differential equation describing the system, and in turn, translate that into a frequency domain version of the same system.

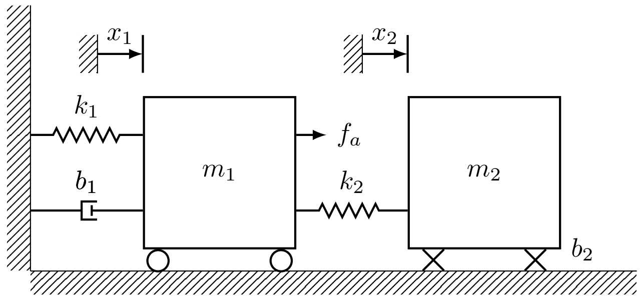

Figure 4: Translational system

2.1 Free body diagram

The first step in finding the differential equation, is to draw a free body diagram of the system in Figure 4. From this we can obtain the differential equations by using Newton's second law, and the d'Alembert principle:

$$\sum f = 0 \tag{5}$$

where the d'Alembert force is included ($m_1\ddot{x}_1$ and $m_2\ddot{x}_2$). The free body diagram can be seen in Figure 5. We use the convention, that arrow pointing to the left (away from the x_1 and x_2 direction) are negative, and arrows pointing right (the same way as x_1 and x_2) are positive. The differential equations are then:

$$-m_1\ddot{x}_1 - k_1x_1 - b\dot{x}_1 + f_a + k_2(x_2 - x_1) = 0 \tag{6}$$
$$-m_2\ddot{x}_2 - k_2(x_2 - x_1) - b_2\dot{x}_2 = 0 \tag{7}$$

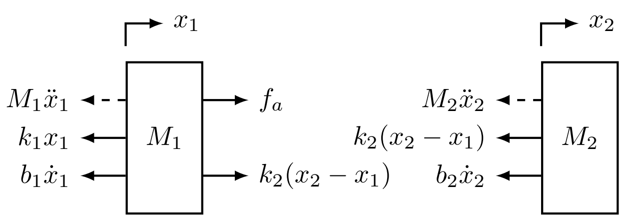

Figure 5: Free body diagram

Laplace transformation is not thoroughly described here, but we are assuming initial conditions to be zero

$$\begin{aligned} -s^2 m_1 X_1(s) - k_1 X_1(s) - s b_1 X_1(s) + F_a(s) \\ + k_2(X_2(s) - X_1(s)) = 0 \end{aligned} \tag{8}$$

$$-s^2 m_2 X_2(s) - k_2(X_2(s) - X_1(s)) - s b_2 X(s) = 0 \tag{9}$$

3 Rotation

The components previously described, can also be described in rotation. Rotation-wise it's often convenient to talk about angle θ instead of position x.

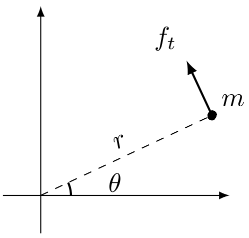

Figure 6: Rotational system

The basic principle can be seen on Figure 6. Here m is a point mass, θ is the angle, r is the distance from the point mass to the rotation center, and f_t is the tangential force.

Newton's second law for a rotating systems is

$$f_t = ma = mr\ddot{\theta} \tag{10}$$

and the torque (moment of force) is

$$\tau = f_t r = mr^2\ddot{\theta} \tag{11}$$

Normally a mass is not in a single point (never, that is). Instead we use the moment of inertia J, to calculate torque

$$\tau = J\ddot{\theta} \tag{12}$$

J can be calculated by integration, but may for most objects be found in books and tables.

The basic components – spring and damper – have rotational characteristics as well as the previously described translational characteristics.

3.1 Spring

The spring torque is dependant on angle, like it was on position before

$$\tau_s = k(\theta_2 - \theta_1) \tag{13}$$

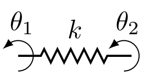

Figure 7: Rotating spring

3.2 Damper

The damper also works quite a lot like the translational equivalent

$$\tau_d = b(\dot{\theta}_2 - \dot{\theta}_1) = b(\omega_2 - \omega_1) \tag{14}$$

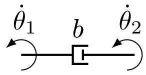

Figure 8: Rotating damper

4 Rotating example system

A rotating system is treated the same way. The system in Figure 9 is seen pictured as a free body diagram in Figure 10.

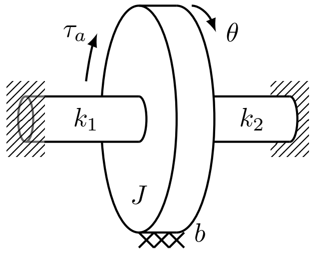

Figure 9: Rotating system

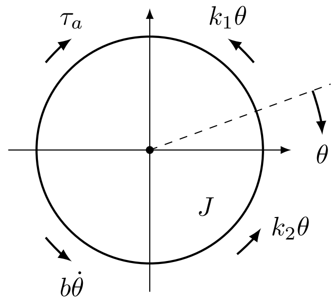

Figure 10: Free body diagram of rotating system.

The differential equation for this system is

$$-J\ddot{\theta} + \tau_a - \theta(k_1 + k_2) - b\dot{\theta} = 0 \tag{15}$$

and thus, the Laplace transformed version is (still assuming initial conditions zero)

$$-s^2 J\Theta(s) + T_a(s) - (k_1 + k_2)\Theta(s) - sb\Theta(s) = 0 \tag{16}$$