Solving roots in binomial complex equations

Søren Nørgaard

29. maj 2012

In this post I will describe how to solve binomial complex equations of the form

$$z^n = a + bi.$$

This is done by first putting the right side on polar form using the following:

$$R = \sqrt{a^2 + b^2}$$
$$\theta = \begin{cases} \tan^{-1}\frac{b}{a} & \text{if } a > 0 \\ \tan^{-1}\frac{b}{a} + \pi & \text{if } a < 0 \\ \frac{\pi}{2} & \text{if } a = 0, b > 0 \\ -\frac{\pi}{2} & \text{if } a = 0, b < 0 \\ \text{random} & \text{if } a = 0, b = 0 \end{cases}.$$

The roots are now found using the following:

$$z = \left(\sqrt[n]{R}\right)_{\frac{\theta}{n} + p\frac{2\pi}{n}} \qquad p = 0, 1, \ldots, n-1$$

where p is altered from 0 to $n-1$, to find all roots.

1 Worked example

Given:

$$z^4 = 3 + 4i$$

Finding R:

$$R = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$
$$R = 5$$

Finding θ (note: $a = 3 \Leftrightarrow a > 0$):

$$\theta = \tan^{-1}\frac{4}{3}$$

The roots are therefore:

$$z_0 = \left(\sqrt[4]{5}\right)_{\frac{3}{4}} \approx 1.495_{0.75}$$
$$z_1 = \left(\sqrt[4]{5}\right)_{\frac{3}{4} + 1\frac{2\pi}{4}} \approx 1.495_{2.32}$$
$$z_2 = \left(\sqrt[4]{5}\right)_{\frac{3}{4} + 2\frac{2\pi}{4}} \approx 1.495_{3.89}$$
$$z_4 = \left(\sqrt[4]{5}\right)_{\frac{3}{4} + 3\frac{2\pi}{4}} \approx 1.495_{5.46}$$

in polar coordinates. To reverse to regular coordinates, just use the following:

$$a = R\cos\theta$$
$$b = R\sin\theta$$

Looking at the complex plane the roots are distributed as follows: