Complex numbers: Roots in polynomials of degree two

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Here is how to solve the roots in polynomials of degree two with complex numbers. After explaining the approach I will go through an example.

We have a polynomial of the form:

$$p(z) = az^2 + bz + c$$

To find the roots of this, we need to know the discriminant D, to know if the roots are complex. This is the case if $D < 0$.

$$D = b^2 - 4ac$$

If $D \geq 0$ the root(s) are real, and may be found using the formula:

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

If the discriminant is negative or complex, we need replace $\sqrt{D}$ with w, which is the solution to the equation $z^2 - D = 0$. This is found by using the formula:

$$w = \sqrt{\frac{r+\alpha}{2}} + i\,\mathrm{sgn}(\beta)\sqrt{\frac{r-\alpha}{2}}$$

Where the discriminant is given as $D = \alpha + \beta i$ and $r = \sqrt{\alpha^2 + \beta^2}$. sgn is $+1$ or -1 depending on the value of β:

$$\mathrm{sgn}(\beta) = \begin{cases} 1, & \text{if } \beta \geq 0, \\ -1, & \text{if } \beta < 0. \end{cases}$$

Now the roots may be found using the formula:

$$z = \frac{-b \pm w}{2a}$$

Using a, b, and c from the original polynomial and w just found.

1 Worked example

Here's a worked example with the polynomial

$$z^2 + 2z - (2 + 4i) = 0$$

First we find the discriminant:

$$D = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot (-(2+4i)) = 12 + 16i$$

As this is a complex determinant, we now need to find w. With the discriminant $D = 12 + 16i$, $r = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$ and $\mathrm{sgn}(\beta) = 1$.

$$w = \sqrt{\frac{20+12}{2}} + i \cdot 1 \cdot \sqrt{\frac{20-12}{2}} = \sqrt{16} + i\sqrt{4} = 4 - 2i$$

This enables us to find the roots for the original polynomial:

$$z = \frac{-b \pm w}{2a} = \frac{-2 \pm (4-2i)}{2 \cdot 1} = \frac{-2}{2} \pm \frac{4-2i}{2} \quad (1)$$

$$z_1 = -1 + 2 - i = 1 - i \quad (2)$$

$$z_2 = -1 - 2 + i = -3 + i \quad (3)$$